The notion of proof and the structure of mathematics

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When I first learn mathematics, I was present with certain component of mathematics - definitions, axioms, and certainly, properties and theorem.

The way we understand it is structural - it is that axioms sets the ground for mathematics to start forming its own - without the ground, you cannot build the house. So in some sense, axioms set everything, into the foundational structure of logic. Otherwise, eventually, without definite axioms, we would throw ourselve into endless recursive logical fallacy - because you can always assume, if there exists no axioms, would the logical unit that you are based on is really "correct"? Are there anything behind it, and behind it, and behind it? You have to stop somewhere, and that's axioms. Somehow, this is oddly in tune with Godel's incompleteness theorem - there's no complete, and consistent theory of mathematics - but there are one powerful enough, and certainly close enough. There just have to be somewhere to start building your house.

In the same way, definitions defines what is there to even build such house. Definition embraces the object it defines, such that to not jump into the second fallacy - of roundabout definition - the definition refers to itself as the definition that it defines. Sounds terrible, doesn't it? So there are actually quite the criteria that we must give to definition - it must, in any case, defines it without mentioning itself, and the definition then would become the construction of the object - there are no "doubt" about such definition. Because A is defined with properties B, and that's it - there's no need to check if an object has property B, would it be A - because it is defined as such. This set apart between **properties** and **definition**, as we can see.

Then, comes the second logical component - theorems. What is theorem, either way? In some way or another, theorem is a statement, that can be proven, or has been proven to be true. Hence, its the complex logical component of the structure of mathematics - it is not self-evident, but is built upon other theorems, of which uses all the definitions and axioms - the object - that the system provides. In one way, theorem is the course of action, the course of deduction, and the course of logical arguments - that leads something, to something, usually facts. Quite an important little guy, one can say.

But what does it mean to prove a theorem? Often time when we learn mathematics, we are thrown a bunch of theorems, lemmas, propositions, and else, sometimes for the sake of defining such. But why are we doing all of it? What gives, only the fact and the truth it brings/ Certainly not. Some facts are useless, but some are useful. So what we are actually doing there, if not just for the basic fact it provides, the truth being verified, and the statement being evaluated?

I dare say, it's **understanding**. Understanding is what we seek along the way. And in some sense, we like to think it as structures. Actually, it helps to think of eberything in a structural manner, of most. Because from it we know why we want understanding than just plain truth. When inspecting certain structures or mathematical objects of interest, we want to understand it. The understanding of such requries one to deduce certain things, from creating certain definitions, and wrap things up by the chain of deductions. At the end of such chain, the revelation is condensed into ”condensed fact” - theorems, that resulted from such manipulation. Those manipulation are entirely valid - they are probes investigating the logical system that they are in. It's just, as we said, not self-evident, because they are hidden from plain sight - the layer, shall we say. There, understanding is important, and it goes for a lot of altitude and actions to theorems - of it, there are requirements, there are interests, there are circumstances, and there are standard. All of which helps us to gain more understanding, again, through the course of action, to then extract more than just the fact that was present itself. Hence, why I said proving the theorem, and the act of proving itself, is more than just extracting the truth.

On the contrary, there are indeed, a lot of times where theorems are there just because of curiosity, or just that we want to know if something is true or not. But why? Why we need it even if perhaps it's useless. The thing is, it might not be. We don't know if it is useless or not in the future that we might not even see. So perhaps, it is better to know it's true, rather than let it be uncertain of lacking efforts.

After all, life is unexpected, and so is mathematics, isn't it?